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CONSTRAINED INEFFICIENCY IN GEI: A GEOMETRIC ARGUMENT

Mario Tirelli

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REDAZIONE:
Dipartimento di Economia
Università degli Studi Roma Tre
Via Silvio D'Amico, 77 - 00145 Roma
Tel. 0039-06-574114655 fax 0039-06-574114771
E-mail: dip_eco@uniroma3.it

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Mario Tirelli*

* *Dipartimento di Economia, Università degli Studi "Roma Tre"*

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MARIO TIRELLI

Department of Economics, University of Rome 3
Via Silvio D'Amico 77, 00145 Rome - Italy
tel: (39) 06 57114735, *fax*: (39) 06 57114610
e-mail: `tirelli@uniroma3.it`

Abstract. In this paper we use global analysis to study the welfare properties of general equilibrium economies with incomplete markets (GEI). Our main result is to show that *constrained Pareto optimal* equilibria are contained in a submanifold of the equilibrium set. This result is explicitly derived for economies with real assets and fixed aggregate resources, of which real numéraire assets are a special case. As a by product of our analysis, we propose an original global parametrization of the equilibrium set that generalizes to incomplete markets the classical one, first, proposed by Lange (1942).

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1. Introduction

Since Radner (1972) there has been a large body of literature studying general equilibrium economies with incomplete markets (GEI). The analysis pioneered by Arrow (1951) and Debreu (1960) on economies with uncertainty and complete markets has been extended in this new direction with contributions addressing traditional issues, such as existence and efficiency of equilibria (see Geanakoplos (1990) and Magill and Shafer (1991) for up to date surveys).

GEI equilibria are typically not Pareto optimal, and may even fail to achieve second-best efficiency. Although, the literature has proposed different notions of *constrained Pareto optimality* (*CPO*), economists often refer to Diamond's (1967), Stiglitz's (1982), and Geanakoplos - Polemarchakis's (1986), as the benchmarks. These notions share the principle that, when implementing an allocation, a central planner faces the same financial constraints of the private sector. This implies that the planner's *attainable* set contains allocations which are *a*) resource-feasible, *b*) achievable through portfolio transfers of the existing assets.

Stiglitz (1982) was the first to provide an argument for constrained Pareto suboptimality of GEI. The intuition behind Stiglitz's result runs as follows. A portfolio redistribution modifies individual income profiles, thereby affecting trading decisions and spot prices. This generates *pecuniary externalities*, which typically have real effects because markets are incomplete. Therefore, a central planner who accounts for these externalities can improve upon the competitive markets allocation of risk.

Geanakoplos and Polemarchakis (1986) and later Geanakoplos et al. (1990), formally, established Stiglitz's result, respectively, in the context of a pure exchange and a production GEI. Precisely, they derived conditions to prove the generic constrained suboptimality of equilibria. The argument used in these classical contributions, and in other papers that followed is essentially based on local analysis. They showed that, for a generic set of economies, equilibria can be locally Pareto improved.[1]

In this paper we propose a different approach to the analysis of the welfare properties of equilibria, based on the global analysis of the equilibrium set. Our main result (theorem 1) is to show that *CPO*-equilibria are exceptional, in the sense that they are contained in a submanifold of the equilibrium set.

To prove this result we proceed in steps. First, we show that the equilibrium set has a fiber bundle structure, which shares most of the properties derived for the equilibrium manifold of a standard Arrow-Debreu (GE) economy in Balasko (1988). Namely, every fiber is a linear submanifold of the equilibrium set, it is uniquely identified by a no-trade equilibrium, and each equilibrium belongs to a fiber only. Since no-trade equilibria need not be (and typically are not) Pareto optimal, the choice of the parametrization cannot simply rely on the welfare weights of the Pareto problem, as first suggested by Lange (1942) for GE. Yet, under a simplifying assumption (assumption 4), we show that this classical approach can be extended to incomplete markets, by defying a fictitious planning problem with an extended system of "welfare weights". This parametrization reduces to Lange's when asset markets are complete.

Second, having recovered the structure of the equilibrium set, we study *CPO* - equilibria fiber-by-fiber. Interestingly, forcing equilibria of each fiber to satisfy the necessary conditions for *CPO* turns out to be equivalent to impose some linear restrictions on the endowments. This readily implies that each fiber contains a linear submanifold (in fact, a sub-fiber) of *CPO* - equilibria. Then, since each equilibrium

[1]See, Magill and Shafer (1991) for a discussion of this approach, or the more recent contribution by Citanna et al.(1998).

belongs to a fiber only, we derive the structure of the set of *CPO* - equilibria by taking the union of these fibers (i.e. of the sub-fibers of the equilibrium manifold). The bundle structure and, in particular, the lower dimensionality of the set of *CPO*-equilibria implies the (generic) constrained suboptimality of equilibria.

The particular fiber-bundle structure we identify says something on the notion of constrained Pareto optimality. Indeed, along each equilibrium fiber, what distinguishes equilibria from *CPO*-equilibria are not their allocations but their transfers. This suggests that the notion of constrained Pareto optimality is one about the efficiency of transfers; i.e. the ability of a competitive market economy to allocate resources relative to an initial distribution.

We complete our analysis for economies with real assets, of which real numéraire assets are a special case. Extensions to nominal assets, and to mixed asset structures are straightforward. This higher level of generality of the analysis comes at the cost of an extra layer of complexity. Indeed, because assets are real, we recover the structure of the equilibrium manifold from that of *pseudo*-equilibria, which is a fiber bundle that retains a vector space structure only locally, on its fibers.

Observe that our geometric argument for constrained suboptimality is substantially different from the one used in the literature. In fact, it does not rely on the characteristics of the parameter space, but directly on the structure of the equilibrium set and on its "size", compared with that of *CPO*-equilibria. This implies that constrained suboptimality can be established without having to appeal to a genericity argument or to impose a specific measure theoretical structure. Obviously, the two concepts of dimensionality and measure can be linked.

Surprisingly, to the best of our knowledge, there are no contributions in the GEI literature analyzing the structure of the set of *CPO*-equilibria. Moreover, there are very few papers studying the structure of the equilibrium set too: Balasko-Cass (1989), Siconolfi and Villanacci (1991), Zhou (1997a,b). The goal of the first two is to analyze the indeterminacy of equilibria in economies with nominal assets, respectively, with variable and fixed aggregate resources. Zhou (1997b) studies the structure of the set of pseudo equilibria and compares its size with the one of equilibria in GEI economies with real assets and aggregate variable resources. The closest to our paper is Zhou (1997a), where for the first time a welfare analysis of GEI is carried out.

Our analysis differs from Zhou's (1997a) in three respects. First, her is only concerned with Pareto optimality (first best) equilibria, while we extend the analysis to constrained Pareto optimality. Second, Zhou assumes variable aggregate resource, we assume fixed. The latest is more natural when it comes to analyze the welfare properties of alternative resource distributions, and it is without loss of generality, since the results trivially generalize to variable resources. Third, because of fixed aggregate resources, our choice of the parametrization is also different. We follow the classical approach outlined by Lange (1942) for GE economies and extend it to GEI.

Our paper is organized as follows. In **section 2**, we provide a few basic notions and definitions. **Section 3** states our main result (theorem 1) and outline the steps we latter follow to prove it. These steps are described in deep in the remaining two sections of the paper. Precisely, in **Section 4**, we globally parameterize the equilibrium set and define its fiber-bundle structure. In **Section 5**, we exploit this structure to establish the welfare properties of equilibria, and prove theorem 1. To improve the exposition, we collect longer or more detailed proofs in an appendix.

2. Basic notions

2.1. Economy and equilibria.

Economy. We consider a two period pure exchange economy, with uncertainty, and finitely many individuals and commodities. There are two dates indexed by 0 and 1. Uncertainty is described by a finite number $S \geq 2$ of possible states of nature in date 1. Including date 0 as one of the states, we use the indexing $s = 0, 1, .., S$, and define $N = (S+1)$. There are a finite number $H \geq 2$ of consumer types, indexed by $h = 1, .., H$. In each state, $L \geq 2$ commodities are available for consumption. A bundle of contingent commodities for h is a vector $x^h = (.., x^h_{sl}, ..)' \in \mathbb{R}^m_+$, where $m = NL$. Without loss of generality, we allow for economies with fixed aggregate resources, $\omega \in \mathbb{R}^m_{++}$, by letting the initial distribution of commodities across agents, $e = \left(.., e^h_{sl}, ..\right)' \in \mathbb{R}^{mH}_{++}$, be an element of the set,[2]

$$\Omega(\omega) = \left\{ e \in \mathbb{R}^{mH}_{++} : \sum_h e^h - \omega = 0 \right\}$$

When ω_s is constant across s, the economy has no aggregate uncertainty.

Each consumer, h, is initially endowed of a vector $e^h = (.., e^h_{sl}, ..)' \in \mathbb{R}^m_+$ of commodities. His preferences are represented by an ordinal utility function $u^h : \mathbb{R}^m \to \mathbb{R}$. Two set of assumptions on endowments and preferences are summarized in the following, and will be maintained throughout the paper.

Assumption 1. *(strictly positive endowments):* $e^h \in \mathbb{R}^m_{++}$

Assumption 2. *(smooth preferences):*[3] $\forall h$, u^h *is* $\mathcal{C}^{r \geq 2}$, *strictly increasing,* $(Du^h(x) \in \mathbb{R}^m_{++}, \forall x \in \mathbb{R}^m_+)$, *strictly concave,* $(bD^2u^h(x)b' < 0, \forall x \in \mathbb{R}^N_{++}, \forall b \in \mathbb{R}^m, b \neq 0,$ *such that* $Du^h(x)\, b' = 0)$; *indifference surfaces are bounded below* $(\forall x^* \in \mathbb{R}^m_{++}, \left\{x \in \mathbb{R}^m_{++} : u^h(x) \geq u^h(x^*)\right\} \subset \mathbb{R}^m_{++})$.

We denote the set of utilities which satisfy assumption 2, $\mathcal{U}$.

An economy is characterized by utilities, endowments, and a set of $J > 0$ (real) assets with payoffs $R \in \mathbb{R}^{SL \times J}$ of date-1 bundles of commodities.

Competitive markets. Commodities and assets are, respectively, traded in spot and asset markets. Commodities are traded at spot prices $p = (.., p_{sl}, ..) \in \mathbb{R}^m_{++}$, where $p_{\mathbf{1}} \in \mathbb{R}^{SL}_{++}$ denotes its date 1 component. Commodity $l = 1$ in state $s = 0$ is the *numéraire*, and its price is normalized to one: $p_{s1} = 1$. We denote the set of normalized prices, $\mathbb{P} \subset \mathbb{R}^{(S+1)L-1}_{++}$.

Assets are traded at prices $q = (.., q^j, ..) \in \mathbb{R}^J$ in date 0, before uncertainty is resolved. Asset j is a real claim yielding a contingent dividend, or financial payoff, $V^j_s = p_s R^j_s$ for all $s > 0$. $V(p_{\mathbf{1}}, R)$ is the $S \times J$ financial matrix whose typical element is V^j_s. Thus, a portfolio $\theta = (.., \theta^j, ..)' \in \mathbb{R}^J$, traded at a market value of $q\theta$ in date 0, pays a financial payoff $V_s(p_{\mathbf{1}}, R)\theta$ in date 1 if state s realizes. We also

[2]At least since Pareto's and Edgeworth's, the assumption of fixed aggregate resources is natural when it comes to analyze the welfare impact of resource redistributions. It generally leads to mathematical complications (see Balasko (1988) chp.V), since it reduces the dimensions along which endowments can be perturbed, thereby strengthening genericity arguments.

[3]We use the standard notation, $Du^h = \left(.., D_{x_{s,l}} u^h(x^h), ..\right) \in \mathbb{R}^m$, where $D_{x_{s,l}} u^h(x^h) = \partial u^h(x^h) / \partial x^h_{s,l}$.

define a price-dividend matrix $W(q,p,R)$, which has the first row equal to $-q$ and the second row-block equal to $V(p_{\mathbf{1}},R)$. Asset markets are incomplete, $J<S$.

Competitive equilibrium. Let us fix (u,ω) and denote an economy by (e,R) with e in $\Omega(\omega)$. At prices (p,q), the budget set of a typical consumer h is,[4]

$$\mathcal{B}(p,q,e^h,R)=\left\{x^h : p\Box\left(x^h-e^h\right)=W(q,p,R)\theta^h,\ \theta^h\in\mathbb{R}^J\right\}$$

The action of h is, respectively, represented by the demand functions for commodities and assets,

$$\begin{aligned} x^h(p,q,e^h,R) &= \left\{x^h : x^h=\arg\max\ u^h(x^h)\ \ s.t.\ x^h\in\mathcal{B}(p,q,e^h,R)\right\} \\ \theta^h\left(p,q,e^h,R\right) &= \left\{\theta^h : p\Box\left(x^h(p,q,e^h,R)-e^h\right)=W(q,p,R)\theta^h\right\}, \end{aligned} \tag{2.1}$$

and spot trades are defined by $z^h(p,q,e^h,R)=x^h(p,q,e^h,R)-e^h$.

Definition 1. ***(Equilibrium)*** *An equilibrium is a tuple (p,q,e,R) such that $\sum_h x^h(p,q,e^h,R)-\omega=0$, $\sum_h\theta^h\left(p,q,e^h,R\right)=0$.*

So far, we have used the asset payoff matrix R to parameterize an economy. This implies that for every spot price p, the columns of $V(p_{\mathbf{1}},R)$ identify an asset span of possibly different dimension. As first noted by Hart (1975), changes in the dimensionality of the asset span may cause discontinuities of individual demands. Thus, to avoid these problems and be on the safe side of smooth economies, one can directly parameterize with respect to the asset span. Formally, this is equivalent to identify an economy with a pair $(e,\mathcal{L})$, where $\mathcal{L}$ is a $J-$plane through the coordinate space $\mathbb{R}^S$, which identifies a space of feasible financial transfers. The collection of such planes defines the Grassmanian manifold, $\mathsf{G}^{J,S}$.

In this "abstract" economy, without loss of generality,[5] we assume that the first individual is financially unconstrained. The Walrasian demand of consumer 1 is a function,

$$g^1\left(p,e^1\right)=\arg\ \max_x\left\{u^1(x) : p\left(x-e^1\right)=0\right\}.$$

The demand of a consumer h is,

$$f^h\left(p,\mathcal{L},e^h\right)=\arg\ \max_x\left\{u^h(x) : \begin{array}{l} p\left(x-e^h\right)=0 \\ p_{\mathbf{1}}\Box\left(x_{\mathbf{1}}-e^h_{\mathbf{1}}\right)\in\mathcal{L}\end{array}\right\} \tag{2.2}$$

for all $h=2,..,H$.

The truncated aggregate excess demand function is $Z(p,\mathcal{L},e)=g^1\left(p,e^h\right)+\sum_{h=2}^H f^h(p,\mathcal{L},e^h)-\omega$, and "abstract"-equilibria, in short **α-equilibria** are defined accordingly,

$$\begin{aligned} \mathcal{E}^\alpha_J &= \left\{(p,\mathcal{L},e)\in\mathbb{P}\times\mathsf{G}^{J,S}\times\Omega : Z(p,\mathcal{L},e)=0\right\}, \\ \mathcal{T}^\alpha_J &= \left\{(p,\mathcal{L},e)\in\mathcal{E}^\alpha_J : g^1(p,pe^1)=e^1,\ f^h(p,\mathcal{L},e^h)=e^h,\forall h\geq 2\right\}. \end{aligned}$$

Observe that $\mathcal{E}^\alpha_J$ is the set of α-equilibria defined for all possible asset span $\mathcal{L}$ in $\mathsf{G}^{J,S}$. $\mathcal{T}^\alpha_J$ is the subset of $\mathcal{E}^\alpha_J$ for which there is no trade.

[4]For any two vectors $x\in\mathbb{R}^S$, $y\in\mathbb{R}^{SL}$, we define $x\Box y=(....,x_s(y_{s1},..y_{sl},..,y_{sL}),...)\in\mathbb{R}^{SL}$.

[5]See remark 4 below.

Definition 2. *(**Pseudo-equilibrium**) A **ψ-equilibrium** is a tuple $(p, \mathcal{L}, e, R)$ such that $(p, \mathcal{L}, e)$ is an element of $\mathcal{E}_J^\alpha$ and the column span of $V(p_{\mathbf{1}}, R)$ is contained in $\mathcal{L}$. $\mathcal{E}_J$ denotes the set of ψ-equilibria, and $\mathcal{T}_J$ the set of no-trade ψ-equilibria.*

The relationship between equilibria in definition 1 and *pseudo-equilibria* is established in Proposition 1,2 in Duffie and Shafer (1985).

2.2. **Constrained Pareto Optimality.** As in Diamond's (1967), Stiglitz's (1982), and Geanakoplos - Polemarchakis's (1986), we assume that the planner's *attainable* set contains allocations which are *a*) resource-feasible, *b*) achievable through portfolio transfers in the existing assets.

More precisely, consider an economy (e, R), in which $e = (e_0, e_{\mathbf{1}})$, and e_0 is any resource-feasible allocation achieved by private agents, eventually through asset or spot trade at R. Centralized transfers are of two types, date-0 lump sum transfers, $t = (.., t^h, ..)$, and portfolio transfers, θ. They have two effects on allocations: date-0 transfers modify e_0 into $e_0 + t$, and portfolio transfers modify $e_{\mathbf{1}}$ into $\widetilde{e}_{\mathbf{1}} = (.., e_{\mathbf{1}}^h + R\theta^h, ..)$. Since final allocations depend on the competitive trade that can occur on spot-markets in date 1, we require they are supported as a date-1 spot-market equilibrium of a date-1 spot-market economy $\widetilde{e}_{\mathbf{1}}$.[6]

This notion of feasibility and the related notion of *CPO* are the ones adopted in Magill and Shafer (1991), with the only minor difference that centralized transfers occur at the end of the first period, when date-0 spot markets are closed.[7]

We now provide formal definitions.

Definition 3. *(**Spot-market equilibrium**) A spot-market equilibrium in s for an economy $\widetilde{e}_s$ is a pair (p_s, x_s) such that*
i) $x_s^h \in \mathbb{R}_{++}^L$ maximizes u^h s.t. $p_s(x_s^h - \widetilde{e}_s^h) = 0$, for all h
ii) $\sum_h (x_s^h - \widetilde{e}_s^h) = 0$.
A date-1 spot-market equilibrium is a spot-market equilibrium in s, for $s = 1, .., S$. The set of spot-markets equilibria in s is E_s and $E_{\mathbf{1}} = \times_{s \geq 1} E_s$ denotes the set of date-1 spot-market equilibria.

Definition 4. *(**Constrained feasible allocations** - CF) A consumption allocation $x = (x_0, x_{\mathbf{1}})$ is constrained feasible (CF) at (e, R) if there exists a $(.., t^h, ..., \theta^h, ..)$ and a $p_{\mathbf{1}}$ such that*
1) $\sum(\theta^h, t^h) = 0$,
2) $x_0^h = e_0^h + t^h$, for all h,
3) $(p_{\mathbf{1}}, x_{\mathbf{1}})$ is a date-1 spot-market equilibrium at $\widetilde{e}_{\mathbf{1}} = (.., e_{\mathbf{1}}^h + R\theta^h, ..)$.

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[6] $\widetilde{e}$ are the "virtual endowments" defined by Magill-Shafer (1991) (for more, see the Remark on pg. 1593).

[7] The notions used in the literature have some minor differences (see Magill and Shafer (1991), chapter 5, for a detailed discussion of this subject). For example Geanakoplos and Polemarchakis (1986) assumed that the planner does not use date-0 transfers, and consider economies in which there is no date-0 consumption. Werner (1991) added date-0 consumption, but maintained the assumption that the planner can only control portfolio transfers, taking some initial date-0 allocation x_0 as given. Magill and Shafer (1991) removed the latter restriction and assumed that the planner uses numéraire transfers in date 0 and allow re-trading both in date-0 and date-1 spot markets. A substantially different notion of feasibility and CPO (known as *weak constrained efficiency*) is instead the one suggested in Grossman (1977), and Grossman and Hart (1979). There, centralized allocations must be supported as a GEI equilibrium; i.e. the central planner intervenes when all markets -including the asset markets- are open.

For simplicity, suppose that date-0 transfers are denominated in the numraire commodity: $t^h = (\tau^h, 0, .., 0) \in \mathbb{R}^L$. Clearly if (p, q, e, R) is an equilibrium of a GEI economy with allocation (x, θ), then $(p_{\mathbf{1}}, x_{\mathbf{1}})$ is a spot-market equilibrium at $\widetilde{e_{\mathbf{1}}}$. Moreover, this allocation x is *CF*, as it can be seen by letting $\tau^h = q\theta^h$ for all h. Finally, since in a date-1 spot-market economy spots are isolated, i.e. the individual budget constraint takes the form $p_s \cdot (x_s^h - \widetilde{e}_s^h) = 0$ in every s, we can re-normalize prices by setting equal to one the price of the first commodity in every spot.

Definition 5. *(**Constrained Pareto optimal allocation** - CPO) A consumption allocation x is a CPO at (e, R) if it is not Pareto dominated by any other allocation that is CF at (e, R).*

Suppose *CPO* allocations exist; to be able to characterizing them using calculus we restrict attention to regular spot-market economies. Formally, we constrain virtual endowments $\widetilde{e}_{\mathbf{1}}$ in definition 4 to be in a generic subset of $\Omega(\omega)$ such that every corresponding spot-market equilibrium $(p_{\mathbf{1}}, x_{\mathbf{1}})$ is a smooth function of $\widetilde{e}_{\mathbf{1}}$.[8]

Consider an equilibrium (p, q, e, R) with an allocation (x, θ) and a financial payoff matrix $V(p, R)$ of full rank. x is supported as a date-1 spot-equilibrium $(p_{\mathbf{1}}, x_{\mathbf{1}})$ of an economy $\widetilde{e_{\mathbf{1}}} = (.., e_{\mathbf{1}}^h + R\theta^h, ..)$. For simplicity, assume the following.

Assumption 3. *(time-separable utilities):* $\forall h,\ u^h(x_0^h, x_{\mathbf{1}}^h) = U_0(x_0^h) + U_{\mathbf{1}}^h(x_{\mathbf{1}}^h)$.

The equilibrium yields a *CPO* allocation, if $\theta = (.., \theta^h, ..)$ solves

$$\begin{array}{ll} Sup_\theta \sum \delta^h v_{\mathbf{1}}^h(P_{\mathbf{1}}, P_{\mathbf{1}} \Box \widetilde{e}_{\mathbf{1}}^h) & s.t. \\ \widetilde{e_{\mathbf{1}}} = (.., e_{\mathbf{1}}^h + R\theta^h, ..),\ \Sigma\theta^h = 0 & \end{array} \tag{2.3}$$

where $v^h(\cdot)$ and δ^h, respectively, denote the date-1 indirect utility function and the welfare weight of h; while $P_{\mathbf{1}}$ is the smooth price functional of a date 1 price-income equilibrium.

The maximization has a solution and its first order conditions are:

$$\left(\delta^h \frac{\partial v_{\mathbf{1}}^h}{\partial m^h} - \delta^1 \frac{\partial v_{\mathbf{1}}^1}{\partial m^1}\right) V(p_{\mathbf{1}}, R)^j + \sum_{h'} \delta^{h'} \frac{dv_{\mathbf{1}}^{h'}}{dP_{\mathbf{1}}} \frac{\partial P_{\mathbf{1}}}{\partial \theta_j^h} = 0, \forall h \geq 2,\ \forall j. \tag{2.4}$$

where, letting m^h denote $p_{\mathbf{1}} \Box \widetilde{e_{\mathbf{1}}}$, the first term on the left hand side is the aggregate income effect, and the second is the aggregate relative price effect. Let $\widehat{\lambda}^h = \frac{\partial v^h}{\partial m^h}$ denote the vector of marginal utility of income for h in (2.3). Using Roy's identity,

$$\begin{aligned} \frac{dv_{\mathbf{1}}^h}{dP_{sl}} &= \frac{\partial v_{\mathbf{1}}^h}{\partial P_{sl}} + \frac{\partial v_{\mathbf{1}}^h}{\partial m^h} \frac{\partial m^h}{\partial P_{sl}} \\ &= -\widehat{\lambda}_s^h x_{sl}^h + \widehat{\lambda}_s^h \widetilde{e}_{sl}^h = -\widehat{\lambda}_s^h \widetilde{z}_{sl}^h \end{aligned}$$

where $\widetilde{z}_{\mathbf{1}}^h = x_{\mathbf{1}}^h - \widetilde{e}_{\mathbf{1}}^h$ for all h, and $\sum \widetilde{z}_{\mathbf{1}}^h = 0$. Letting $\lambda_s^h = \delta^h \widehat{\lambda}_s^h$, we can rewrite (2.4) as,

$$\left(\lambda^h - \lambda^1\right) V^j - \sum_{h',s} \sum_{l \geq 2} \lambda_s^{h'} \widetilde{z}_{sl}^{h'} \frac{\partial P_{sl}}{\partial \theta_j^h} = 0,\ for\ all\ h \geq 2,\ all\ j$$

[8]This same procedure can be found in Magill and Shafer (1991) p.1596. The existence of such a subset of regular economies can be shown using standard techniques: once it is established that $E_{\mathbf{1}}$ is a manifold, one can define a smooth natural projection, mapping this set onto the set of endowments (see the discussion in chapter V, Balasko (1988)).

Next, recall that to study the welfare properties of an equilibrium, we evaluate (2.4) at an equilibrium. Let $\delta^h = 1/\lambda_0^h$; $\lambda^h = (1, \lambda_{\mathbf{1}}^h)$ is the $S-$vector of normalized state prices of h at an equilibrium.[9] Individual first order optimality conditions imply no-arbitrage: $\left(\lambda^h - \lambda^1\right) V^j = 0$ for all h, j. Following Stiglitz (1982), one can test for the *CPO* of an equilibrium allocation by checking if it satisfies the $(H-1)J$ equations:[10]

$$\sum_h \left(\lambda^h \Box \widetilde{z}_{\mathbf{1}}^h\right) D_\theta P_{\mathbf{1}} = 0 \tag{2.5}$$

This last condition says that a *CPO*-equilibrium is one such that there do not exist, feasible, portfolios redistributions that can induce indirect welfare effects; with "indirect effects" meaning effects propagating through changes in relative spot prices (i.e. "pecuniary externalities").

Finally, it is useful to recall the exact form of $D_\theta P_{\mathbf{1}}$. First of all, $D_\theta P_{\mathbf{1}}$ is well defined if $D_P Z_{\mathbf{1}}(p, m)$ is invertible,

$$D_\theta P_{\mathbf{1}} = -(D_P Z_{\mathbf{1}})^{-1} D_\theta Z_{\mathbf{1}}$$

This, in our context, says that $D_\theta P_{\mathbf{1}} = 0$ if, at least locally, agents have the same "propensities to consume",[11] $\kappa_{\mathbf{1}}^h = D_{m^h} \widetilde{x}_{\mathbf{1}}^h$:

$$\begin{aligned} D_\theta Z_{\mathbf{1}} &= (..., D_{\theta_j^h} Z_{\mathbf{1}}, ...) \in \mathbb{R}^{S(L-1)\times J(H-1)} \\ D_{\theta_j^h} Z_{\mathbf{1}} &= \left(D_{m^h} \widetilde{x}^h - D_{m^1} \widetilde{x}^1\right) V^j \end{aligned}$$

Next, recall that the Slutzky matrix is the inverse of the Jacobian of the individual demand system. Since here we are concerned with a spot-market individual demand, let us consider the individual demand of consumer h in date 1, at prices and income $(p_{\mathbf{1}}, m^h)$, $\widetilde{x}_{\mathbf{1}}^h(p_{\mathbf{1}}, m^h)$. Then,

$$\begin{pmatrix} D^2 U_{\mathbf{1}}^h & -p_{\mathbf{1}}' \\ -p_{\mathbf{1}} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \mathsf{S}_{\mathbf{1}}^h & -\kappa_{\mathbf{1}}^{h\prime} \\ -\kappa_{\mathbf{1}}^h & -a_{\mathbf{1}}^h \end{pmatrix} \tag{2.6}$$

where $\mathsf{S}_{\mathbf{1}}^h$ is the matrix of substitution effects. Simple computations yield,

$$\mathsf{S}_{\mathbf{1}}^h = \left(D^2 U_{\mathbf{1}}^h\right)^{-1} \left[D^2 U_{\mathbf{1}}^h - p_{\mathbf{1}}' \left(p_{\mathbf{1}} (D^2 U_{\mathbf{1}}^h)^{-1} p_{\mathbf{1}}'\right) p_{\mathbf{1}}\right] \left(D^2 U_{\mathbf{1}}^h\right)^{-1}$$

$$\kappa_{\mathbf{1}}^h = \left(D^2 U_{\mathbf{1}}^h\right)^{-1} p_{\mathbf{1}}' \left(p_{\mathbf{1}} (D^2 U_{\mathbf{1}}^h)^{-1} p_{\mathbf{1}}'\right)^{-1} \tag{2.7}$$

where $U_{\mathbf{1}}^h$ is evaluated at $\widetilde{x}_{\mathbf{1}}^h = \widetilde{x}_{\mathbf{1}}^h(p_{\mathbf{1}}, m^h)$.

Rapping up, $D_\theta Z_{\mathbf{1}}$ is a function of the date-1 spot-equilibrium allocation $x_{\mathbf{1}}$ and of the financial matrix V. This leads to the following important remark.

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[9] Consider the representation of a Planner problem in which δ^h is the Lagrange multipliers associated to the constraint, $v_{\mathbf{1}}^h(P, m) \geq \overline{v}_{\mathbf{1}}^h$, and take $\overline{v}_{\mathbf{1}}^h$ to be the utility level achieved at date 1, in a competitive equilibrium. Then, letting $\delta^h = 1/\lambda_0^h$ is equivalent to say that there exist welfare weights such that the original equilibrium satisfies CPO necessary conditions; this is in the spirit of the *I welfare theorem.* If, instead, we fix welfare weights, and we ask if an allocation that satisfies necessary conditions for CPO can be achieved at equilibrium, then we need to introduce date-0 transfers. The latter is the perspective of the *II welfare theorem.*

[10] $D_\theta P_{\mathbf{1}} = (..., D_{\theta_j^h} P_{\mathbf{1}}, ...) \in \mathbb{R}^{S(L-1)\times(H-1)J}$, has typical column vector $D_{\theta_j^h} P_{\mathbf{1}} = (\partial P_{12}/\partial \theta_j^h, ..., \partial P_{SL}/\partial \theta_j^h)'$.

[11] This case requires utilities to behave, at least locally, as ones having a Gorman form.

Remark 1. *For any fixed consumption allocation, portfolio allocation, financial and real payoff matrices,* (x, θ, V, R)*, condition (2.5) is a linear function of date-*1 *initial endowments* $e_{\mathbf{1}}$*:*

$$\sum_h \lambda^h(x^h)\Box(x^h_{\mathbf{1}} - e^h_{\mathbf{1}} - R\theta^h)D_\theta P_{\mathbf{1}}(x_{\mathbf{1}}, V) = 0 \tag{2.8}$$

is linear in $e_{\mathbf{1}}$*.*

3. Main result

In this section we argue that *CPO* equilibria are "exceptional" in the sense that we make precise as follows.

Theorem 1. *CPO - equilibria are contained in a lower dimensional submanifold of the equilibrium set.*

Our proof of theorem 1 uses a few interesting preliminary results, and is deferred to section 5. Precisely, to prove this theorem, we first derive a global parametrization of the equilibrium set. This parametrization is intuitive and allows as to characterize the equilibrium set as a fiber bundle. In the process, we show that this characterization shares most of the properties presented for Arrow-Debreu (GE) equilibria in Balasko (1988):

- *every fiber is contained in the equilibrium set;*
- *every fiber is a linear submanifold of the equilibrium set*;
- *every fiber contains only one (and is therefore identified by a) no-trade equilibrium;*
- *every equilibrium belongs to one fiber only.*
- *the set of equilibria is obtained by taking the disjoint union of its fibers*

Moreover, if asset markets are complete the equilibrium set is equivalent to the set of equilibria of economies with contingent markets, reproducing Balasko's characterization of GE as a special case.

Then, we prove our main result by showing that,

- *each fibber contains a linear submanifold of* CPO*-equilibria with the property that all these equilibria have the same allocation but different levels of trade. This linear submanifold is a* ***sub-fiber*** *of the equilibrium set.*
- *the set of CPO-equilibria is a submanifold of the equilibrium set, obtained by taking the disjoint union of its sub-fibers.*

The basic idea behind our characterization is as follows. In GE economies, no-trade equilibria are Pareto optima (*PO*), i.e. the I Welfare Theorem applies. Therefore, as first noticed by Lange (1942), no-trade GE can be simply recovered using the solutions of a *PO* problem; the parametrization of the set of *PO* is indeed a global parametrization of the set of no-trade GE. In extending this logic to economies with incomplete markets, we run into two obstacles. First, no-trade GEI are, typically, not *PO*. Second, *CPO* equilibria, generically, entail some trade across agents (i.e. they do not belong to the no-trade equilibrium set). We outrode the first obstacle by showing that no-trade equilibria can actually be represented as solutions of the following "modified" planner's problem: let utilities be state-separable (see assumption 4 below) and x solve,

$$Max_x \sum_h \sum_s \chi^h_s U^h_s(x^h_s) \qquad \text{s.t.} \qquad \Sigma x^h \leq \omega$$

at "welfare weights" $\chi = (1, ., \chi_s^h, ..) \in \mathbb{R}^{H(S+1)}$. Differently from Pareto optima, in this problem welfare weights are state-contingent. Moreover, we show that in GEI with $(S-J)$ degrees of market incompleteness, χ lives in a $(H-1)+(H-1)(S-J)$- dimensional set. If markets are complete, and $J=S$, this parametrization is the one used for no-trade GE. However as J decreases, falling below S, the dimensionality of the parameter space increases. In the limit -when $J=0$- the dimension of the parametrization is the one corresponding to the equilibrium set of a $(S+1)-$ spot-market economy, in which each spot is indeed an isolated Arrow-Debreu economy.

Equipped with this parametrization of no-trade, the global structure of the equilibrium set is easily derived. For every no-trade equilibrium, $(\overline{p}, \overline{\mathcal{L}}, \overline{x}, \overline{R})$, one can identify the set of ψ-equilibria $(\overline{p}, \overline{\mathcal{L}}, e, \overline{R})$ with active trade $z = \overline{x} - e$. This boils down to considering the set of economies parameterized by initial endowments e such that $(\overline{x})$ is budget-feasible and satisfies markets clearing at $(\overline{p}, \overline{\mathcal{L}}, e, \overline{R})$. This subset of $\Omega(\omega)$ has dimension $n + SJL$, where $n = (H-1)(m-(S-J+1))$. Therefore, for fixed aggregate resources, one finds that the ψ-equilibrium manifold is described by taking the disjoint union of its fibers, where each fiber is identified by a no-trade ψ-equilibrium:[12]

$$\mathcal{E}_J \cong \mathcal{T}_J \times \mathbb{R}^n_{++} \cong \Omega(\omega) \times \mathbb{R}^{SJL}$$

Next, we still have to overcome the second obstacle. Even though we have been able to parameterize no-trade equilibria, we still have to show that this is what we effectively need to parameterize the set of *CPO*-equilibria. Recall that, unlike for Pareto optima, this is not obvious because *CPO*-equilibria do typically entail some trade. Yet, this trading activity are easier to represent if *CPO*-equilibria are analyzed fiber-by-fiber. Restrict attention to the generic set of economies, $\Omega^* \times \mathcal{R}^*$, such that ψ- equilibria are characterized by a financial matrices, V, of full rank. An equilibrium fiber is identified by a no-trade allocation x. Along this fiber equilibria share the same allocation, x, and the same triplet of prices, real payoff matrix, financial matrix, (p, R, V), but different trades, $(z_0, z_{\mathbf{1}}, \theta)$. A necessary condition for an equilibrium (p, e, R) with allocation (x, θ), to be *CPO* is that portfolio transfers θ satisfy conditions (2.5):

$$\sum_h \lambda^h(x^h)\Box(x^h_{\mathbf{1}} - e^h_{\mathbf{1}} - R\theta^h)D_\theta P_{\mathbf{1}} = 0$$

This is convenient since now the last expression is linear in date-1 endowments (see remark 1 above). Therefore, along each fiber *CPO*-restrictions impose $c \leq (H-1)J < n$ independent linear constraints on the endowments, on top of the equilibrium ones mentioned above.[13]

Interestingly, what distinguishes equilibria from *CPO*-equilibria -along each equilibrium fiber- are not their allocations but their transfers. This suggests that the notion of constrained Pareto optimality is one about the efficiency of the system of competitive trades, rather than of allocations. Put it differently, it is akin to evaluate how well competitive markets allocate resources relative to an initial distribution.

[12] $\cong$ denotes equivalence up to a homeomorphism.

[13] We shall argue that $c > 0$, by noticing that $c = 0$ occurs only for a null subset of economies parameterized by endowments and utilities.

4. The structure of equilibria

We, first, proceed by studying the global structure of α-equilibria. The results obtained, and in particular the parametrization of α-equilibria, are then used to pin down the fiber bundle structure of the set of pseudo-equilibria. Since these bundles have $\mathsf{G}^{J,S}$ as a base space, we begin by stating some well known properties of Grassmanians.

4.1. **Grassmanians.** $\mathsf{G}^{J,S}$ is the smooth compact manifold of J-planes in $\mathbb{R}^S$. Let Y denote the manifold of $(S-J)\times J$ matrices of rank $S-J$. Any Y in Y induces some element $\mathcal{L}$ of $\mathsf{G}^{J,S}$: $\mathcal{L}=\{y\in\mathbb{R}^S : Yy=0\}$. Define the equivalence relation $\sim$ on Y as, $Y\sim Y'$ if and only if there exists a non-singular, square, matrix B such that $Y'=BY$. We identify $\mathsf{G}^{J,S}$ with the quotient space $\mathsf{Y}/\sim$.

To make explicit the differentiable structure of $\mathsf{G}^{J,S}$, we now describe its atlas.[14] Let σ be a permutation of $\{1,..,S\}$, Σ the set of all such permutations, and π_σ the $S\times S$ permutation matrix associated to σ. For every $\mathcal{L}$ in $\mathsf{G}^{J,S}$ there exists a $\sigma\in\Sigma$, and a unique local coordinate system $A\in\mathbb{R}^{S-J\times J}$ of $\mathcal{L}$: $A=\psi_\sigma(\mathcal{L})$ where ψ_σ is a *homeomorphism* of $W_\sigma=\{\mathcal{L}\in\mathsf{G}^{J,S} : \exists\, A\in\mathbb{R}^{(S-J)\times J}\ s.t.\ (I_{S-J}\mid A)\pi_\sigma\in\mathcal{L}\}$ onto $\mathbb{R}^{J(S-J)}$. $\{W_\sigma,\psi_\sigma\}_{\sigma\in\Sigma}$ is a smooth atlas for $\mathsf{G}^{J,S}$. Moreover, the union of W_σ over Σ defines an open cover of $\mathsf{G}^{J,S}$, implying that $\mathsf{G}^{J,S}$ is a compact smooth manifold.

Over the same base space $\mathsf{G}^{J,S}$, we define two vector bundles:[15] the *canonical vector bundle* $\upsilon=\{\mathcal{L},y\in\mathsf{G}^{J,S}\times\mathbb{R}^S : y\in\mathcal{L}\}$, and its orthogonal complement $\upsilon^\perp=\{\mathcal{L},y\in\mathsf{G}^{J,S}\times\mathbb{R}^S : y\perp\mathcal{L}\}$. $\mathsf{G}^{J,S}$ *splits* as a Whitney sum of $\upsilon,\upsilon^\perp$: $\upsilon\oplus\upsilon^\perp=\mathbb{R}^S$.[16] That is, $\mathbb{R}^S$ is (locally) equivalent to a vector space spanned by (y,y') where, for all $\mathcal{L}\in\mathsf{G}^{J,S}$, $(\mathcal{L},y)\in\upsilon$ and $(\mathcal{L},y')\in\upsilon^\perp$.

4.2. **Parametrization.** Our choice of parametrization for the equilibria is better understood by recalling a classical result in GE (see Lange 1942).

Let welfare weights $\delta=(\delta^1,..,\delta^H)$ be in the interior of the $(H-1)$-simplex, Δ^{H-1}_{++}. $x\gg 0$ is a GE allocation if and only if it maximizes

$$\Sigma\delta^h u^h(x^h)\qquad s.t.\qquad \Sigma x^h\leq\omega \tag{4.1}$$

Using strict concavity and monotonicity of utilities, the latter holds if and only if x solves the system of first order conditions, $\Sigma x^h=\omega$ and $\delta^h Du^h-\rho=0$, for some $\rho\in\mathbb{R}^m_{++}$. This implies that $x=(x^1,..,x^h,..)\in\Omega(\omega)$ solves

$$\nabla^h(x^h)-\nabla^1(x^1)\ =\ 0,\ \forall h\geq 2$$

where $\nabla^h=\frac{1}{D_{01}u^h}Du^h$. The first line is a well known optimality condition: at a Pareto optimal allocation agents have identical marginal rates of substitution. For expositional reasons, we write this condition as

$$\mu^h(x)=\left(....,\frac{\nabla^h_{s,l}-\nabla^1_{s,l}}{\nabla^1_{s,l}},....\right)=0,\ \forall(h,l)\geq(2,1), \tag{4.2}$$

[14] See Fact 3 in Duffie and Shafer (1985), and Hirsch (1976) for definitions and references.

[15] A fiber bundle with vector space structure on fibers is a *vector bundle*. See chapter 4 in Hirsch (1976) for a detailed exposition on vector bundles on $\mathsf{G}^{J,S}$.

[16] $\oplus$ denotes the Whitney sum. This operates as a direct sum across the elements of the fibers of a vector bundle.

where μ^h is a vector valued function whose values are in $\mathbb{R}^S$.

To check that the solution $x \gg 0$ of the Pareto problem (4.1) is indeed an equilibrium, it basically amounts to verify that each of its components, x^h, satisfies consumer h problem at $(\frac{1}{\rho_{01}}\rho, x^h)$. Clearly, since we are endowing consumers with demand allocations, a no-trade equilibrium arises. The converse is also immediate, and provides a calculus argument to establish the I Welfare Theorem. Therefore, it is easily understood that the no-trade equilibrium manifold of an Arrow-Debreu economy is equivalent to Δ^{H-1}_{++}, independently on the number of commodities and states.

Since GE, contingent markets, economies are substantially equivalent to complete markets GEI, we claim that the latter parametrization applies to these economies too. Moreover, interpreting μ as a measure of individuals disagreement on *state-prices*, we verify that -at a Pareto optimal allocation- this is zero, i.e. agents share the same evaluations of future contingent income profiles.

Next, consider the opposite extreme case of a GEI economy in which financial markets are totally incomplete. This economy is equivalent to the $(S+1)$-copy of a standard GE economy with L commodities. Thus, indexing welfare weights with respect to the states, we conjecture that the manifold of no-trade equilibria of a GEI economy with no asset markets is equivalent to $\times_{S+1}\Delta^{H-1}_{++}$.

The remaining question is how to parameterize equilibria for "non-trivial" GEI economies, in which markets are partially incomplete. The following assumption simplifies the characterization of (no-trade) equilibria, and it is necessary to extend Lange's.

Assumption 4. *(State-separable utilities):* $\forall h,\ u^h(x^h) = \sum_s U^h_s(x^h_s)$.

We conjecture that $x \gg 0$ is a α-equilibrium allocation if and only if it maximizes

$$\sum_h \sum_s \chi^h_s U^h_s(x^h_s) \qquad \text{s.t.} \qquad \Sigma x^h \leq \omega \tag{4.3}$$

at "welfare weights" $\chi = (\chi_0, .., \chi_s, ..)$,
$\chi_0 = (..., \delta^h, ...) \in \Delta^{H-1}_{++}, \quad \chi_s = (..., \delta^h/(1+\mu^h_s), ...),\ \forall s \geq 1$
$\mu^1 = 0, \quad \mu = (\mu^h)_{h>1}, \quad (\mathcal{L}, \mu) \in \mathbb{M}_J = \{\mathcal{L}, \mu : \mu^h \in \mathcal{L}^\perp,\ \mu^h_s \neq -1,\ \forall (h,s) > (1,0)\}$.

To get an intuition of why our conjecture is true, we can proceed as for GE economies. To verify that a solution $x \gg 0$ of (4.3) is an α-equilibrium, we have to check that each of its components x^h satisfies consumer h problem at $(\frac{1}{\rho_{01}}\rho, \mathcal{L}, x^h)$. Indeed, let x be an interior optimum at $(p, \mathcal{L}, e)$, then there exist multipliers, $(\lambda^h, \gamma^h) \in \mathbb{R}_{++} \times \mathbb{R}^{S-J}$, such that,

$$\begin{aligned} &\nabla^1(x^1) = (p_0, p_{\mathbf{1}}) \\ &\nabla^h(x^h) = (p_0, p_{\mathbf{1}}) + [0, \tfrac{\gamma^h}{\lambda^h}(I \mid \psi_\sigma(\mathcal{L}))\pi_\sigma \square p_{\mathbf{1}}],\ \forall h \geq 2,\ \sigma \text{ } s.t.\ \mathcal{L} \in W_\sigma \end{aligned}$$

Letting $\mu^h = \frac{\gamma^h}{\lambda^h}(I \mid \psi_\sigma(\mathcal{L}))\pi_\sigma$, the equivalent of (4.2) is

$$\mu^h(x) = \left(...., \frac{\nabla^h_{s,l} - \nabla^1_{s,l}}{\nabla^1_{s,l}}, ... \right) \in \mathcal{L}^\perp,\ \forall (h,l) \geq (2,1)$$

Thus, expanding our parametrization to μ allows to describe a much reacher set of equilibria than the one containing just Pareto optima. Let us denote by $\mathbb{M}_J$ the parameter space of typical element $(\mathcal{L}, \mu)$ defined for an economy with a J-dimensional asset span. $\mathbb{M}_J$ summarizes all the "relevant" information concerning asset markets, encompassing "trivial" GEI too. In fact, if markets are complete ($S = J$), then $\mathcal{L}^{\perp} = \{0\}$ implying $\mu = 0$; therefore $\mu = 0$, and $\dim \mathbb{M}_S = 0$: consumers' normalized gradients are all equal to p. At the other extreme, if there are no assets ($J = 0$), $\mathcal{L}^{\perp} = \mathbb{R}^S$, $\dim \mathbb{M}_0 = (H-1)S+J(S-J)$. Thus, if $(H-1) > S$, consumers' state prices may span $\mathbb{R}^S$ (i.e. there may be "maximal disagreement" among consumers).

The next lemma establishes the global structure of $\mathbb{M}_J$ as a vector bundle over $\mathsf{G}^{J,S}$; where $\mathbb{M}_J$ is regarded as the total topological space, $\mathsf{G}^{J,S}$ as the base space, and α_J as the projective map.

Lemma 1. *$\mathbb{M}_J$ is a vector bundle over $\mathsf{G}^{J,S}$, $\mathbb{M}_J \cong (H-1)\upsilon^{\perp}$.*[17] *Its projective map, α_J, identifies a unique $\mathcal{L}$ for each $(\mathcal{L}, \mu)$ in $\mathbb{M}_J$.*

Proof: see the Appendix.

Since $\mathbb{M}_J$ has a vector space structure on its fibers, locally (on its base space) it is also a manifold of dimension $(H-1)(S-J)$.

4.3. **The structure of no-trade α-equilibria.** To derive the structure of the equilibrium set we will appeal to Lemma 3.2.1 in Balasko (1988), limiting our proofs to the definition of the required diffeomorphisms:

Lemma 2. *(Lemma 3.2.1 in Balasko (1988)) Let $\tau : X \to Y$, and $\phi : Y \to X$, be two smooth mappings between smooth manifolds such that the composition $\tau \circ \phi : Y \to Y$ is the identity mapping. Then, the set $Z = \phi(Y)$ is a smooth submanifold of X diffeomorphic to Y.*

The definition of a parametrization, Y, is an essential step in the derivation of our next results.

Proposition 1. *$\mathcal{T}^{\alpha}_J$, is a manifold diffeomorphic to $\Delta^{H-1}_{++} \times \mathbb{M}_J$. Moreover, as a fiber bundle over $\mathsf{G}^{J,S}$, $\mathcal{T}^{\alpha}_J \cong \varepsilon^{H-1} \oplus (H-1)\upsilon^{\perp}$.* [18]

Proof: see the Appendix.

To apply lemma 2, we let $Y = \Delta^{H-1}_{++} \times \mathbb{M}_J$, and $X = \mathcal{E}^{\alpha}_J$. For the time being, we will assume that $\mathcal{E}^{\alpha}_J$ is a smooth manifold;[19] this is established in proposition 2 below. Moreover, we let $\tau^{\mathcal{T}^{\alpha}} : \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega \to \Delta^{H-1}_{++} \times \mathbb{M}_J$,

$$\tau^{\mathcal{T}^{\alpha}}(p, \mathcal{L}, e) = \left(\left(\frac{D_{01}u^1(x^1)}{D_{01}u^h(x^h)} \right)_{h\geq 2}, \mathcal{L}, \mu^h(x)_{h\geq 2} \right)$$

where, if $\tau^{\mathcal{T}^{\alpha}}$ is restricted to $\mathcal{E}^{\alpha}_J$, $(x^1, (x^h)_{h\geq 2}) = (g^1(p, e^1), (f^h(p, \mathcal{L}, e^h))_{h\geq 2})$ are equilibrium allocations.

Next, we let $\phi^{\mathcal{T}^{\alpha}} : \Delta^{H-1}_{++} \times \mathbb{M}_J \to \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega$, be such that

$$\phi^{\mathcal{T}^{\alpha}}(\delta, \mathcal{L}, \mu) = \left(\nabla u^1(x^1), \alpha_J(\mathcal{L}, \mu), x\right) = (p, \mathcal{L}, e)\,,$$

[17] $\cong$ denotes an *homeomorphism* relationship.

[18] ε^n_B denotes the *trivial vector bundle* $(B \times \mathbb{R}^n, B, \alpha)$ with base space B; a fiber bundle that has a global (instead of a local) vector space structure. We drop the scripts n, B, from ε, when they clearly emerge from the context.

[19] This is well known, and is established in Fact 9 of Duffie and Shafer (1985).

where we take x to be the maximizer of problem (4.3), with "welfare weights" χ defined accordingly.

Remark 2. *(The structure of $\mathcal{T}_J^\alpha$) When $J \in \{0, S\}$ the set of no-trade α-equilibria has a (globally) trivial (vector space) structure. Using $\mathcal{T}_S^\alpha$ to denote the no-trade complete markets version of $\mathcal{T}_J^\alpha$, it is easy to see that, $\mathcal{T}_S^\alpha \cong \Delta_{++}^{H-1}$.*[20] *This is the case in which $\mu = 0$, and consumer gradients are collinear. On the other extreme, when asset markets are totally incomplete, we have $\mathcal{T}_0^\alpha \cong \times_s \Delta_{++}^{H-1}$. In all intermediate cases, $0 < J < S$, $\mathcal{T}_J^\alpha \cong \mathcal{T}_S^\alpha \oplus (H-1)\upsilon^\perp$; namely $\mathcal{T}_J^\alpha$ retains a vector space structure only locally on $\mathsf{G}^{J,S}$.*

4.4. **The structure of α-equilibria with active trade.** Next, consider the set $\mathcal{E}_J^\alpha$ of abstract-equilibria in which consumers trade an initial set of resources (or endowments). The new parameter space will then be enlarged by the dimensionality of the space of relevant spot trade opportunities, $n \equiv (H-1)(m-(S-J+1))$.

Proposition 2. *$\mathcal{E}_J^\alpha$ is a smooth manifold diffeomorphic to $\Omega(\omega) \times \mathbb{R}^{(S-J)J}$. Moreover, as a fiber bundle over $\mathsf{G}^{J,S}$, $\mathcal{E}_J^\alpha \cong \varepsilon^{(H-1)+n} \oplus (H-1)\upsilon^\perp$.*

Proof: see the Appendix.

In order to apply Lemma 2, let $X = \Delta_{++}^{H-1} \times \mathbb{M}_J \times \mathbb{R}_{++}^n$, and $Y = \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega$. Moreover, define $\tau^{\mathcal{E}^\alpha} : \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega \to \mathbb{R}_{++}^{H-1} \times \mathbb{M}_J \times \mathbb{R}_{++}^n$, such that

$$\tau^{\mathcal{E}^\alpha}(p, \mathcal{L}, e) = \left(\tau^{\mathcal{T}^\alpha}(p, \mathcal{L}, e), proj_{\mathbb{R}_{++}^n}\Omega\right)$$

Next, let $\phi^{\mathcal{E}^\alpha} : \Delta_{++}^{H-1} \times \mathbb{M}_J \times \widehat{\Omega} \to \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega$, be such that
$\phi^{\mathcal{E}^\alpha}(\delta, \mathcal{L}, \mu, \widehat{e}) = \left(\nabla u^1(x(\delta, \mathcal{L}, \mu)), \alpha_J(\mathcal{L}, \mu), e\right)$,
and $\widehat{\Omega}$ be a section of $\Omega(\omega)$, defined as follows. Observe that $(p, \mathcal{L})$ in the image of $\phi^{\mathcal{E}^\alpha}$ support a no-trade equilibrium. To achieve an equilibrium with active trade it suffices to restrict endowments to satisfy individual budget balance and overall feasibility. This implies that we should restrict e to satisfy:

$$\begin{array}{lll} & p(x^h - e^h) = 0 & \text{all } h \geq 2 \\ (4.4) & (I_{S-J} \mid \psi_\sigma(\mathcal{L}))\pi_\sigma(p_{\mathbf{1}} \square (x_{\mathbf{1}}^h - e_{\mathbf{1}}^h))^T = 0 & \text{all } h \geq 2,\ \sigma \ s.t.\ \mathcal{L} \in W_\sigma \\ & e \in \Omega(\omega) & \end{array}$$

evaluated at $x = x(\delta, \mathcal{L}, \mu)$. This can be done, for example, by constraining the following $d \equiv (H-1) + (H-1)(S-J) + (S+1)L$ endowments, $(e_{01}^h)_{h\geq 1}, ((e_{s1}^h)_{s=1}^{S-J})_{h\geq 2}, (e_{sl}^1)_{(s,l)\geq(0,1)}$, to satisfy (4.4). Then, $\widehat{\Omega} \subseteq \mathbb{R}_{++}^n$ with $n = Hm - d = (H-1)(m-(S-J+1))$ is the section of $\Omega(\omega)$ containing all but the constrained endowments.

[20] See Balasko (1988), section 3.3 for details on the structure of no-trade α-equilibria when markets are complete.

4.5. **Economies with real assets and pseudo-equilibria.** Observe, first, that an α-equilibrium $(p, \mathcal{L}, e)$ of an abstract economy $(\mathcal{L}, e)$ is indeed a ψ-equilibrium of each of those real assets economies (R, e) such that R induces a financial market structure $V(p_{\mathbf{1}}, R)$ whose column span is contained in $\mathcal{L}$. That is, loosely speaking, for every $(p, \mathcal{L}, e)$ there are multiple real assets economies whose financial market possibilities, at $p_{\mathbf{1}}$, are feasible in an abstract-economy $(\mathcal{L}, e)$. It helps to formalize this concept by defining a set of *financial market possibilities* with real assets. When, restricted to $\mathbb{P} \times \mathsf{G}^{J,S} \times \mathbb{R}^{SLJ}$, this set is

$$\mathcal{M}_J = \{p, \mathcal{L}, R : (I \mid \psi_\sigma(\mathcal{L}))\pi_\sigma V\,(p_{\mathbf{1}} R) = 0, \sigma \in \Sigma \ s.t.\, \mathcal{L} \in W_\sigma\}\,.$$

The set of ψ-equilibria can now be redefined accordingly,

$$\mathcal{E}_J = \{p, \mathcal{L}, e, R : (p, \mathcal{L}, e) \in \mathcal{E}^\alpha_J,\ (p, \mathcal{L}, R) \in \mathcal{M}_J)\}\,.$$

Lemma 3. *$\mathcal{M}_J$ is a manifold diffeomorphic to $\mathbb{P} \times \mathbb{R}^{SJL}$. Moreover as a fiber bundle over $\mathsf{G}^{J,S}$, $\mathcal{M}_J \cong \varepsilon^{NL-1+SJ(L-1)} \oplus J\upsilon$.*

Proof: see the Appendix.

Proposition 3. *$\mathcal{E}_J$ is a smooth manifold diffeomorphic to $\Omega(\omega) \times \mathbb{R}^{SJL}$. Moreover, as a fiber bundle over $\mathsf{G}^{J,S}$, $\mathcal{E}_J \cong \mathcal{E}^\alpha_J \oplus \mathcal{M}_J$.*

Proof: see the Appendix.

In order to apply lemma 2, first, let V denote a $S{\times}J$-matrix and $\mathbb{N}_J = \{\mathcal{L}, \mu, V : \mu^h \in \mathcal{L}^\perp, < V > \subseteq \mathcal{L}\}$, where $< V >$ denotes the space spanned by the columns of V. Along exactly the same lines of lemma 1, we can show that $\mathbb{N}_J$ is a vector bundle over $\mathsf{G}^{J,S}$, $\mathbb{N}_J \cong (H-1)\upsilon^\perp \oplus J\upsilon$, and a smooth manifold diffeomorphic to $\mathsf{G}^{J,S} \times \mathbb{R}^{(H-1)(S-J)+J^2} \simeq \mathbb{R}^{(H-1)(S-J)+SJ}$.

Next, let $X = \mathbb{P} \times \mathbb{N}_J \times \widehat{\Omega} \times \mathbb{R}^{SJ(L-1)}$ and $Y = \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega(\omega) \times \mathbb{R}^{SJL}$. Define the map $\phi^{\mathcal{E}} : \mathbb{P} \times \mathbb{N}_J \times \widehat{\Omega} \times \mathbb{R}^{SJ(L-1)} \to \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega(\omega) \times \mathbb{R}^{SJL}$ such that $\phi^{\mathcal{E}}(p, (\mathcal{L}, \mu, V), \widehat{e}, r(-1)) = (p, \mathcal{L}, e, R)$, with $(p, \mathcal{L}, e) \in Im(\phi^{\mathcal{E}^\alpha}(p, \mathcal{L}, \mu, \widehat{e}))$, and R that satisfies the following restriction: its components referring to commodity $l = 1$ satisfy $V(p_{\mathbf{1}}, R) = V$, all its remaining components are equal to the corresponding entries of $r(-1)$.[21] Finally, let $\tau^{\mathcal{E}} : \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega(\omega) \times \mathbb{R}^{SJL} \to \mathbb{P} \times \mathbb{N}_J \times \widehat{\Omega} \times \mathbb{R}^{SJ(L-1)}$ be such that $\tau^{\mathcal{E}}(p, \mathcal{L}, e, R) = (p, \mathcal{L}, \mu, \widehat{e})$ where $(p, \mathcal{L}, \mu, \widehat{e}) = \tau^{\mathcal{E}^\alpha}(p, \mathcal{L}, e)$, $V = V(p_{\mathbf{1}}, R)$.

Then, denote by E_s the Arrow-Debreu equilibrium manifold in the state s economy, and by E the GE manifold of an economy with a complete set of contingent markets. We can state the following corollary without proof.

Corollary 1. *$\mathcal{E}_J$ has a (globally) trivial (vector space) structure if and only if $J \in \{0, S\}$.*
1) the set of equilibria is the Cartesian product of the set of the $S{+}1$ spot-market equilibria, $\mathcal{E}_0 = \times_{s=0}^{S} E_s$ iff markets are "totally" incomplete, $J = 0$.
2) The set of equilibria is equivalent to the set of Arrow-Debreu equilibria, $\mathcal{E}_S = E$, iff markets are complete, $J = S$.
If $0 < J < S$, $\mathcal{E}_J$ retains a trivial structure only locally, on $\mathsf{G}^{J,S}$.

[21] $r = (r(1), r(-1)) \in \mathbb{R}^{SLJ}$ represent the whole vector of real payoffs in R, where $r(1) = (r^1_{11}, ..., r^J_{S1}) \in \mathbb{R}^{SJ}$ refers to the commodity $l = 1$ and $r(-1) \in \mathbb{R}^{SJ(L-1)}$ to all $l > 1$.

Finally, appealing to remark 2, a second corollary applies.

Corollary 2. *The set of no-trade ψ-equilibria is a smooth manifold diffeomorphic to* $\mathbb{R}^{(H-1)(S-J+1)+SJL} \times \mathsf{G}^{S,J}$. *Moreover, as a fiber bundle over* $\mathsf{G}^{J,S}$, *it is homeomorphic to* $\mathcal{T}_J^{\alpha} \oplus \mathcal{M}_J$.

Remark 3. *(Equilibria) Theorem 2 in Duffie and Shafer (1985) establishes that there is an open set of economies* $\Omega^* \times \mathcal{R}^* \subseteq \Omega \times \mathbb{R}^{SLJ}$ *with null complement such that an equilibrium with real asset exists for every* $(e, R) \in \Omega^* \times \mathcal{R}^*$. *The proof of this theorem relies on the fact that for a generic set of economies,* ψ*-equilibria are characterized by a financial matrix* V *of full column rank* J. *Indeed, this implies that for every* ψ*-equilibrium* $(p, \mathcal{L}, e, R)$ *and associated matrix* $V(p_{\mathbf{1}}, R)$ *of full rank, there exists a pair* (θ, q) *such that* (p, q, e, R) *is an equilibrium with portfolios* θ *(see their Proposition 1,2). See also Zhou (1997b), who derives this result comparing the size of the two sets of equilibria.*

Remark 4. *(Symmetric equilibria) In this section we have considered an economy in which the first consumer is financially unconstrained. This goes with the name of "Cass trick" and has been used repeatedly in the GEI literature. It simplifies and is without loss of generality, since for most of the analysis it suffices to know that the equilibrium manifolds of the economies with symmetric agents and with a financially unconstrained one are equivalent, at least up to a diffeomorphism. For the case of real assets we refer to Magill-Shafer (1991), p.1534.*

4.6. **The fiber bundle structure of ψ-equilibria.** We are now going to define the fiber bundle structure of $\mathcal{E}_J$. This will then be used to study the welfare properties of equilibria in the next section.

Definition 6. *A fiber associated to* $\xi = (\delta, (\mathcal{L}, \mu, V), R(-1)) \in \Xi = \Delta_{++}^{H-1} \times \mathbb{N}_J \times \mathbb{R}^{SJ(L-1)}$ *is a set,* $\mathcal{F}_\xi$, *of typical element* $(p, \mathcal{L}, e, R) \in \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega(\omega) \times \mathbb{R}^{SJL}$: $\mathcal{F}_\xi$ *is the inverse image of* $\{\xi\} \times \widehat{\Omega}$ *by* $\tau^{\mathcal{E}}$.

Definition 6 is better understood by looking back at the above results. In particular, proposition 3 and corollary 2 imply that $\mathcal{E}_J$ is homeomorphic to $\varepsilon^n \oplus \mathcal{T}_J^{\alpha} \oplus \mathcal{M}_J$. This, loosely speaking, says that we can fix no-trade ψ-equilibrium, and generate an n -dimensional set of equilibria with active trade, in $\mathcal{E}_J$. Precisely, recall that $\tau^{\mathcal{E}} : \mathcal{E}_J \to \Xi \times \widehat{\Omega}$, associates to every equilibrium $(p, \mathcal{L}, e, R)$ a unique ξ, where $\widehat{\Omega} \subseteq \mathbb{R}^n$. Moreover, since each ξ identifies a no-trade,

$$\mathcal{F}_\xi = (\tau^{\mathcal{E}})^{-1}(\{\xi\} \times \widehat{\Omega}) \tag{4.5}$$

Notice that a fiber associated to ξ contains those equilibria in $\mathcal{E}_J$ which: **i**) are compatible with a fixed pair $(p, \mathcal{L})$, a fixed equilibrium allocation (the corresponding no-trade equilibrium allocation), and a fixed real and financial assets structure (R, V); **ii**) have different level of endowments $\widehat{e}$ in $\widehat{\Omega}$, and thus of trades z.

Our fibers have a few interesting properties, which are analogous to those established for Arrow-Debreu equilibria in Balasko (1988).

- *Every fiber is a subset of* $\mathcal{E}_J$.

In fact, by definition, $(\tau^{\mathcal{E}})^{-1}(\xi) = (p, \mathcal{L}, e, R)$ is an element of $\mathcal{E}_J$.

- *Every fiber is a linear submanifold of* $\mathcal{E}_J$, *of dimension* n.

This also follows by definition (equation (4.5)).

- *Every fiber contains only one no-trade ψ-equilibrium.*

This follows from the structure of no-trade, which are diffeomorphic to $\mathbb{R}^{(H-1)(S-J+1)+SJL} \times \mathsf{G}^{S,J}$ (see corollary 2).

- *Every ψ-equilibrium belongs to one fiber only.*

This follows from definition 6: each fiber is identified by a unique no trade equilibrium. This does also explain the following,

- *Fibers can be "glued" together by letting ξ vary on Ξ: $\mathcal{E}_J \cong \dot{\cup}\mathcal{F}_\xi$.*[22]

5. Welfare properties of equilibria

Let $\mathcal{E}_J^*$ denote the ψ-equilibrium manifold restricted to $\Omega^* \times \mathcal{R}^*$. By defying the diffeomorphism $\tau^{\mathcal{E}^*}$ accordingly, we obtain a fiber-bundle structure of equilibria that shares the same properties of the one described in the last subsection. Moreover, by remark 3, now each equilibrium fiber contains ψ-equilibria with a full rank financial matrix V and well defined portfolio transfers θ.

We now characterize *CPO*-equilibria fiber-by-fiber. Let ξ identify a fiber of $\mathcal{E}_J^*$. Along this fiber equilibria share the same allocation (the no-trade allocation identified by ξ), the same prices, the same real payoff matrix and financial matrix, (x, p, R, V) respectively. Since V is of full rank, each equilibrium with date-1 spot trade $(.., z_{\mathbf{1}}^h, ..) = (..., x_{\mathbf{1}}^h - e_{\mathbf{1}}^h, ...)$ has portfolios $\theta = (.., \theta^h, ..)$ such that θ^h satisfies $p_{\mathbf{1}} \square z_{\mathbf{1}}^h = V\theta^h$ for all $h \geq 2$ and $\theta^1 = \Sigma_{h>1}\theta^h$. A necessary condition for a ψ-equilibrium $(p, \mathcal{L}, e, R)$ with allocation (x, θ), on a fiber ξ, to be *CPO* is that portfolio transfers θ satisfy conditions (2.5), or (2.8). These conditions can be rewritten as:

$$\sum_h \mu^h(x) \square (x_{\mathbf{1}}^h - e_{\mathbf{1}}^h - R\theta^h) D_\theta P_{\mathbf{1}}(x_{\mathbf{1}}, V) = 0 \tag{5.1}$$

where $P_{\mathbf{1}}(\cdot)$ is the equilibrium spot-prices function. As noted earlier in remark 1, from a simple inspection of (5.1) one sees that this is linear in $e_{\mathbf{1}}$. Therefore, along a fiber, we identify *CPO*-equilibria by superimposing the condition that $e_{\mathbf{1}}$ satisfies (5.1), evaluated at the equilibrium variables.[23] This implies the following.

- *Each fiber contains a linear submanifold of* CPO*-equilibria with the property that all these equilibria have the same allocation but different levels of trade. This linear submanifold is a* ***sub-fiber*** *of the equilibrium set, $\mathcal{F}_{\xi,\underline{e}}$, where $\underline{e} \in \mathbb{R}^c$, $0 < c \leq (H-1)J$.*
- *Clearly, $\mathcal{F}_{\xi,\underline{e}}$ is a (lower dimensional) submanifold of $\mathcal{F}_\xi$, of codimension c.*[24]

Finally, we can use the fact that each *CPO*- equilibrium belongs to one fiber, and conclude that

- *there exists a set of CPO-equilibria that is identified by taking the disjoint union of sub-fibers, $\mathcal{E}_J^{CPO} \subseteq \dot{\cup}_\xi \mathcal{F}_{\xi,\underline{e}}$.*

---

[22] $\dot{\cup}$ denotes the disjoint union of sets.

[23] Since the planner's problem needs not be convex, fibers may identify equilibria which fail to satisfy second order conditions for CPO.

[24] To compute its actual dimension, suppose that $c = (H-1)J$, i.e. *CPO*-conditions (5.1) are independent equations, then $dim\mathcal{F}_{\xi,\underline{e}} = n - (H-1)J = (H-1)((S+1)L - (S-J+1)) - (H-1)J$, which is equal $(H-1)(S+1)(L-1)$.

These last three items rely on the fact that $c > 0$; i.e. the conditions in (5.1) are not trivially satisfied at equilibrium. There are a few well known sufficient conditions that can be invoked to show that (5.1) holds at an equilibrium (see Stiglitz (1982) or Geanakoplos and Polemarchakis (1986)): (i) identical, individual, state-prices ($\mu^h = 0$, for all h); (ii) no trade; (iii) asset redistribution have no price effects ($D_\theta P_{\mathbf{1}} = 0$). These conditions are non-generic in the sense that they hold only for a null subset of economies. More precisely, (i) does not typically hold in an equilibrium of a GEI economy; while properties (ii) and (iii) are non-generic for competitive economies regardless their markets are complete or incomplete. For the first, observe that the set of equilibria satisfying (i) is Pareto optimal, and thus it is diffeomorphic to the set of no-trade GE, $\mathcal{T}_S$. Based on what argued in remark 2 for α-equilibria, we know that $\mathcal{T}_S$ is a submanifold of $\mathcal{T}_J$ with positive codimension. Thus, by Sard's theorem,[25] there exists a generic set of economies for which equilibria do not satisfy (i). An analogous dimensionality argument can be used for (ii): we showed that $\mathcal{T}_J$ is a submanifold of $\mathcal{E}_J$ (see corollary 2). As for (iii), this holds if agents have identical marginal propensities to consume (see the discussion in section 2.2). Inspection of equation (2.7), defining the marginal propensity to consume, hints that one can use local perturbations of the Hessian of utilities to locally control the propensities to consume. These perturbations can be designed such as to change the Hessian of a consumer without affecting his utility gradient, and thus his consumption choices. This argument has been extensively used in the GEI literature to show that there exist an open and dense set of utility functions such that the economies parameterized by these utilities generate equilibria in which (iii) does not hold.[26]

We summarize our discussion in the following.

Remark 5. *There is an open and dense set of economies* $\Omega^* \times \mathcal{R}^* \times \mathcal{U}^* \subseteq \Omega \times \mathbb{R}^{SLJ} \times \mathcal{U}$ *such that for every one of such economies equilibria imply that* $c > 0$.

We can now exploit the above results to prove our main result.

Proof of Theorem 1: We just need to use the fact that the set of CPO*-equilibria,* $\mathcal{E}_J^{CPO}$, *is contained in a submanifold of the equilibrium set,* $\dot{\cup}_\xi \mathcal{F}_{\xi,\underline{e}}$. ■

Finally, the fiber bundle structure of $\mathcal{E}_J$ *can be used to establish the, generic, constrained inefficiency of equilibria. This comes again as a straightforward application of Sard's theorem, implying that the set of economies with CPO-equilibria is null.*

Appendix

Proof of Lemma 1: We limit our proof to the construction of a local isomorphism. Fix $\mathcal{L} \in W_\sigma$, *with* $A = \psi_\sigma(\mathcal{L})$ *identifying the local coordinate system of* $\mathcal{L}$. *Define,* $h_\sigma(\Gamma) = \Gamma(I_{S-J} \mid \psi_\sigma(\mathcal{L}))\pi_\sigma$, *where* $\Gamma \in \mathbb{R}^{(H-1)\times(S-J)}$ *and* $\Gamma\psi(\mathcal{L})$ *have elements different from* -1. $\mu \in Imh_\sigma(\Gamma)$ *and* $(\mathcal{L}, \mu)$ *is an element of* $\mathbb{M}_J$. h_σ *is injective:* $\Gamma(I_{S-J} \mid A)\pi_\sigma = \Gamma'(I_{S-J} \mid A)\pi_\sigma$ *implies* $\Gamma = \Gamma'$. h_σ *is surjective: every* $\mu \in Imh_\sigma(\Gamma)$ *is, by definition, made of* $(H-1)$*-vectors in* $\mathcal{L}^\perp$*; hence there exists a non*

[25] See, for example, Guillemin and Pollack (1974), p. 205.

[26] See the seminal paper by Geanakoplos and Polemarchakis (1986), and particularly their proof of proposition 5; or Magill and Shafer (1991) p.1600.

trivial $\Gamma \in \mathbb{R}^{H-1 \times S-J}$ *such that* $\mu = \Gamma(I_{S-J} \mid A)\pi_\sigma$*; using a conformable partition,* $\mu = (\mu_{S-J} \mid \mu_J)$, $\Gamma = \mu_{S-J}$. ■

Observe that $x \in (g^1(p, e^1), .., f^h(p, \mathcal{L}, e^h), ..)$ *if and only if there exist Lagrange multipliers,* $(\lambda^h, \gamma^h) \in \mathbb{R}_{++} \times \mathbb{R}^{S-J} \backslash \{0\}$*, such that,*

$$(D_0 u^1(x^1), D_{\mathbf{1}} u^1(x^1)) = \lambda^1(p_0, p_{\mathbf{1}}) \\ p(x - e^1) = 0 \tag{.2}$$

$$\big(D_0 u^h(x), D_{\mathbf{1}} u^h(x)\big) = \lambda^h\,(p_0, p_{\mathbf{1}}) + (0, \gamma^h (I \mid \psi_\sigma(\mathcal{L}))\pi_\sigma \Box p_{\mathbf{1}}), \forall h \geq 2 \\ p(x - e^h) = 0, \forall h \geq 2 \\ (I \mid \psi_\sigma(\mathcal{L}))\pi_\sigma p_{\mathbf{1}} \Box (x_{\mathbf{1}} - e^h_{\mathbf{1}}) = 0, \forall h \geq 2 \tag{.3}$$

where σ *is taken s.t.* $\mathcal{L} \in W_\sigma$.

Proof of Proposition 1: *Let us start with* τ*. When* $\tau^{\mathcal{T}^\alpha}$ *is restricted to* $\mathcal{E}^\alpha_J$*,* x *are equilibrium allocations, and* $\delta^h = D_{01}u^1(x)/D_{01}u^h(x) \in \mathbb{R}_{++}$*, for all* $h \geq 2$*. To show that* $(\mathcal{L}, \mu) \in \mathbb{M}_J$*, observe that* $x^h \in f^h(p, \mathcal{L}, e)$*, by individual first order conditions, implies that there exists a* $\lambda^h \in \mathbb{R}_{++}$, $\gamma^h \in \mathbb{R}^{S-J}$ *(possibly zero), such that* $\nabla_{\mathbf{1}} u^h(x) = (\mathbf{1}_S + \frac{\gamma^h}{\lambda^h}(I_{S-J} \mid \psi_\sigma(\mathcal{L}))\pi_\sigma \Box p_{\mathbf{1}}$*. By definition of* $\tau^{\mathcal{T}^\alpha}$, $\mu^h = \frac{\gamma^h}{\lambda^h}(I_{S-J} \mid \psi_\sigma(\mathcal{L}))\pi_\sigma$ *for all* $h \geq 2$*, implying* $(\mathcal{L}, (.., \mu^h, ..)) \in (H-1)v^\perp$*. Finally, assuming that* $\mathcal{E}^\alpha_J$ *is a smooth manifold and, exploiting the differentiable structure of* $\mathbb{M}_J$*, we conclude that* $\tau^{\mathcal{T}^\alpha}$ *has smooth coordinates, i.e. it is smooth.*

Next, consider ϕ*. To show that* $\phi^{\mathcal{T}^\alpha}$ *is well defined, and smooth, it suffices to prove that the set of solutions to (4.3), is -respectively- nonempty and its elements,* $x(\delta, \mathcal{L}, \mu)$*, are smooth functions on* $\Delta^{H-1}_{++} \times \mathbb{M}_J$*. Nonemptiness follows from the fact that (4.3) is the maximization of a continuous function on a compact set. Because utilities are strictly concave, for every system of welfare weights* χ *defined by some* $(\delta, \mathcal{L}, \mu) \in \Delta^{H-1}_{++} \times \mathbb{M}_J$*, this maximization has a unique solution. We establish that* $x(\delta, \mathcal{L}, \mu)$ *is smooth in lemma 4, below.*

We then argue that the image of $\phi^{\mathcal{T}^\alpha}$ *is* $\mathcal{T}^\alpha_J$ *using the results of the next two items.*

- **:** *Im* $\phi^{\mathcal{T}^\alpha} \subseteq \mathcal{T}^\alpha_J$*. Consider the first order (necessary and sufficient) conditions of (4.3): for all* $h \geq 2$

$$\delta^h D_0 u^h\left(x^h\right) = D_0 u^1\left(x^1\right) = \overline{p}_0 \\ D_{\mathbf{1}} u^1\left(x^1\right) = \overline{p}_{\mathbf{1}} \\ \delta^h D_{\mathbf{1}} u^h\left(x^h\right) = \rho_{\mathbf{1}}(\mu^h, \overline{p}_{\mathbf{1}}) = ((1 + \mu^h_s)\overline{p}_s)_{s \geq 1} \tag{.4}$$

where $\overline{p} \in \mathbb{R}^m_{++}$ *is the vector of Lagrange multipliers of the resource constraints; and markets clear,* $x(\delta, \mathcal{L}, \mu) \in \Omega(\omega)$*. Notice that* $p = \nabla u^1(x^1(\delta, \mathcal{L}, \mu)) = \frac{1}{\overline{p}_{01}}\overline{p} \in \mathbb{P}$*, and* $e = x(\delta, \mathcal{L}, \mu)$*. We are left to check that, at* $(p, \mathcal{L}, e) = \big(\nabla u^1(x^1), \alpha_J(\mathcal{L}, \mu), x\big)$*, agents optimize; i.e. individual first order conditions in (.2), (.3) hold, respectively, for* $h = 1$ *and all* $h \geq 2$*. With some re-writing, the latter are,*

$$\tfrac{1}{\lambda^h} D_0 u^h\left(x^h\right) = \tfrac{1}{\lambda^1} D_0 u^1\left(x^1\right) = p_0 \\ \tfrac{1}{\lambda^1} D_{\mathbf{1}} u^1\left(x^1\right) = p_{\mathbf{1}} \\ \tfrac{1}{\lambda^h} D_{\mathbf{1}} u^h\left(x^h\right) = (\mathbf{1}_S + \tfrac{\gamma^h}{\lambda^h}(I \mid \psi_\sigma(\mathcal{L}))\pi_\sigma \Box p_{\mathbf{1}} \tag{.5}$$

where $(\lambda^h)^{-1} = \delta^h$ *for all* $h \geq 2$, *and* $\lambda^1 = D_{01}u^1\left(x^1(\delta, \mathcal{L}, \mu)\right)$. *Letting* $\overline{\gamma}^h = \frac{\gamma^h}{\lambda^h}$ *for all* $h \geq 2$, *we make use of the mapping* h_σ *(defined in the proof of lemma 1), granting that* $(\overline{\gamma}, \mathcal{L}) = h_\sigma^{-1}(\mathcal{L}, \mu)$ *is uniquely defined at* $(\mathcal{L}, \mu)$.

: $Im\,\phi^{\mathcal{T}^\alpha} \supseteq \mathcal{T}_J^\alpha$. *Let us show that* $\phi^{\mathcal{T}^\alpha} \circ \tau^{\mathcal{T}^\alpha} = id_{\mathcal{T}_J^\alpha}$ *when* $\tau^{\mathcal{T}^\alpha}$ *is restricted to* $\mathcal{T}_J^\alpha$. *If* $(p, \mathcal{L}, e) \in \mathcal{E}_J^\alpha$, *and the equilibrium allocation is* $x = e$, *then individual first order conditions, (.5), hold at* e, *and so do (.4) at* $(\delta, \mathcal{L}, \mu) = \tau(p, \mathcal{L}, e)$; *because* $\sum_h e^h = \omega$, e *is a solution to (4.3) at* $(\delta, \mathcal{L}, \mu) = \tau(p, \mathcal{L}, e)$.

By lemma 2, $\phi^{\mathcal{T}^\alpha}$ *defines the desired diffeomorphism, and when* $\tau^{\mathcal{T}^\alpha}$ *is restricted to* $\mathcal{T}_J^\alpha$, $\tau^{\mathcal{T}^\alpha} \circ \phi^{\mathcal{T}^\alpha} = id_{\mathbb{R}_{++}^{H-1} \times \mathbb{M}_J}$.

Finally, the fiber bundle structure of $\mathcal{T}_J^\alpha$ *follows immediately from the structure of* $\mathbb{M}_J$. ■

Lemma 4. $x(\delta, \mathcal{L}, \mu)$ *is a smooth function on* $\mathbb{M}_J \times \Delta_{++}^{H-1}$

Proof : *The proof applies the implicit function theorem, and mimics the one used to show that the interior maximum of a Pareto problem is a smooth function of the parameters. The only, minor, difference here is that problem (4.3) has welfare weights which are state-contingent. Indeed, in analogy with a Pareto maximum problem, every solution of (4.3) is interior. Then, first order (necessary and sufficient) conditions of (4.3) are,*

1 $\chi_s^h D_{sl} U_s^h(x^h) - D_{sl} U_s^1(x^1) = 0, h \geq 2, s \geq 0, l \geq 1$

2 $\sum_h x_{sl}^h - \overline{\omega} = 0,\ s \geq 0, l \geq 1,$

We denote this system as $F(x; \chi, \overline{\omega}) = 0$, *and* $\mathcal{K}$ *the set of solutions of (4.3). Since* $\mathcal{K} = ImF^{-1}(0)$, 0 *is a regular value of* F *if its Jacobian,* $D_x F$, *is of full rank. Since* u^h *are* $\mathcal{C}^{r\geq 2}$, F *and* $x(\delta, \mathcal{L}, \mu)$ *are* $\mathcal{C}^{r \geq 1}$, *by the Implicit Function Theorem. Computing* $D_x F$:

$$
\begin{matrix}
-D^2U^1 & \ddots & & & & 0 \\
 & & \ddots & & & \\
\vdots & & & \chi_s^h D_s^2 U_s^h & & \\
 & & & & \ddots & \\
-D^2U^1 & 0 & & & & \ddots \\
I_m & I_m & & I_m & & I_m
\end{matrix} \tag{.6}
$$

Since column operations do not affect the rank of (.6), subtract the first column block from the h^{th}, *for* $h = 2, .., H$; *then, move the resulting* H^{th} *row block (a block–row vector of typical element* I_m *in the first* H *blocks) to the top row block. The following matrix representation is obtained,*

$$
\begin{pmatrix} I_m & 0 \\ * & O \end{pmatrix} \tag{.7}
$$

$$
\begin{aligned}
O &= \begin{pmatrix} G^2 & D^2U^1 & \cdots & D^2U^1 \\ D^2U^1 & \ddots & & \vdots \\ \vdots & & \ddots & D^2U^1 \\ D^2U^1 & \cdots & D^2U^1 & G^H \end{pmatrix} \\
G^h &= \begin{pmatrix} \chi_1^2 D_1^2 U_1^h + D_1^2 U_1^1 & & 0 \\ & \ddots & \\ 0 & & \chi_S^h D_S^2 U_S^h + D_S^2 U_S^1 \end{pmatrix}
\end{aligned}
$$

and $G_s^h \in \mathbb{R}^{L\times L}$. We are going to show that O is of full rank, because otherwise negative definiteness of D^2u^h (in assumption 2) would be contradicted. For $r^1 \in \mathbb{R}^{(H-1)m}$, r^1O has typical (L-vector) element, $\chi_s^h r_{s,h}^1 D_s^2 U_s^h + \left(\sum_{h=2}^H r_{s,h}^1\right) D_s^2 U_s^1$. Post multiplying the latter by ${r^1}_{s,h}^T$, and summing over $h \geq 2$, yields

$$
\sum_{h=2}^H \chi_s^h \left(r_{s,h}^1 \left(D_s^2 U_s^h\right) {r^1}_{s,h}^T \right) + \left(\sum_{h=2}^H r_{s,h}^1\right) D_s^2 U_s^1 \left(\sum_{h=2}^H r_{s,h}^1\right)^T \tag{.8}
$$

By assumption 2), the two terms in the latter expression are negative, and so is their sum. Hence, (.8) is equal to zero if and only if $r_{s,h}^1 = 0$, for all s and all $h \geq 2$. ■

Proof of Proposition 2: Recall that, by definition, $\phi^{\mathcal{E}^\alpha}(\delta, \mathcal{L}, \mu, \widehat{e}) = \left(\nabla u^1(x^1), \alpha_J(\mathcal{L}, \mu), e\right) = (p, \mathcal{L}, e)$ with e satisfying the restrictions in (4.4) at $(p, \mathcal{L}, \widehat{e})$, and $\tau^{\mathcal{E}^\alpha}(p, \mathcal{L}, e) = (\tau^{\mathcal{T}^\alpha}(p, \mathcal{L}, e), proj_{\mathbb{R}^n_{++}}\Omega)$. Clearly, $\phi^{\mathcal{E}^\alpha}$ and $\tau^{\mathcal{E}^\alpha}$ have smooth coordinates and are defined between smooth manifolds. Moreover, by straightforward computations, they satisfy $\tau^{\mathcal{E}^\alpha} \circ \phi^{\mathcal{E}^\alpha} = id_{\Delta_{++}^{H-1} \times \mathbb{M}_J \times \widehat{\Omega}}$. To apply lemma 2, we need to show that $Im\phi^{\mathcal{E}^\alpha} = \mathcal{E}_J^\alpha$. The endowments restrictions in (4.4) imply $Im\phi^{\mathcal{E}^\alpha} \subseteq \mathcal{E}_J^\alpha$; while $Im\phi^{\mathcal{E}^\alpha} \supseteq \mathcal{E}_J^\alpha$ follows from observing that $\phi^{\mathcal{E}^\alpha} \circ \tau^{\mathcal{E}^\alpha} = id_{\mathcal{E}^\alpha}$. Therefore, $\mathcal{E}_J^\alpha = \phi^{\mathcal{E}^\alpha}(\Delta_{++}^{H-1} \times \mathbb{M}_J \times \widehat{\Omega})$, is a smooth submanifold of $\mathbb{P} \times \mathsf{G}^{J,S} \times \Omega$ diffeomorphic to $\Delta_{++}^{H-1} \times \mathbb{M}_J \times \widehat{\Omega}$ through $\phi^{\mathcal{E}^\alpha}$. ■

Proof of Lemma 3:[27] *We proceed by showing that $\mathcal{M}_J$ and $\varepsilon^{NL-1+SJ(L-1)} \oplus J\upsilon$ are homeomorphic. Then, since the latter is a vector bundle, it has a vector space structure on fibers (over $\mathsf{G}^{J,S}$); therefore for $\mathcal{M}_J$ to be a manifold of dimension $(NL-1) + SJL = (NL-1) + (S-J)J + J^2 + SJ(L-1)$, we need to show that such an homeomorphism is in fact a diffeomorphism (i.e. it is smooth and has a smooth inverse). To define the desired homeomorphism, let $r = (r(1), r(-1)) \in \mathbb{R}^{SLJ}$ represent the whole vector of real payoffs in R, where $r(1) = \left(r_{11}^1, ..., r_{S1}^J\right) \in \mathbb{R}^{SJ}$ refers to commodity $l = 1$ and $r(-1) \in \mathbb{R}^{SJ(L-1)}$ to all $l > 1$. Define $\tau^{\mathcal{M}}(p, \mathcal{L}, R) = \left(p, \mathcal{L}, \left(V^1, .., V^J\right), r(-1)\right)$ such that $V^j \in \mathcal{L}$ for all j. That is $\left(\mathcal{L}, \left(V^1, .., V^J\right)\right)$ is an element of the J-copy of the canonical vector bundle υ, $J\upsilon$, whose dimension, over $\mathsf{G}^{J,S}$, is J^2.*

Next, let $\left(\mathcal{L}, \left(V^1, .., V^J\right)\right) \in J\upsilon$, $\phi^{\mathcal{M}}\left(p, \mathcal{L}, \left(V^1, .., V^J\right), r(-1)\right) = (p, \mathcal{L}, R)$ is defined such that R is formed by $r = (r(1), r(-1))$ with $r_s^j(1) = R_{s,1}^j = \frac{1}{p_{s,1}}\left(V_s^j - \sum_{l>1} p_{sl} r_{sl}^j(-1)\right)$ for all $(s, j) \geq (1, 1)$.

Thus, $\tau^{\mathcal{M}}$ is an homeomorphism between $\mathcal{M}_J$ and $\varepsilon^{NL-1+SJ(L-1)} \oplus J\upsilon$, with its inverse, ${\tau^{\mathcal{M}}}^{-1} = \phi^{\mathcal{M}}$.

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[27] This proof is substantially the same of Theorem 6 in Zhou (1997a).

Finally, to show that $\tau^{\mathcal{M}}$ *is also a diffeomorphism it suffices to show that* 0 *is a regular value of* $(I \mid \psi_\sigma(\mathcal{L}))\pi_\sigma V(p_{\mathbf{1}}R)$*, or that its Jacobian with respect to* R *is of full row rank* $(S-J)J$ *(as in Fact 7, Duffie and Shafer (1985)).* ■

Proof of Proposition 3: *Recall that the two mappings,* $\phi^{\mathcal{E}}(p,(\mathcal{L},\mu,V),\widehat{e},r(-1)) = (p,\mathcal{L},e,R)$ *and* $\tau^{\mathcal{E}}(p,\mathcal{L},e,R) = (p,\mathcal{L},\mu,\widehat{e})$ *are defined between smooth manifolds, and have smooth coordinates. Moreover, by straightforward computations,* $\tau^{\mathcal{E}} \circ \phi^{\mathcal{E}} = id_{\Delta_{++}^{H-1}\times\mathbb{N}_J\times\widehat{\Omega}\times\mathbb{R}^{SJ(L-1)}}$*. To apply lemma 2, we need to show that* $Im(\phi^{\mathcal{E}}) = \mathcal{E}_J$*. The restrictions imposed on the image of* $\phi^{\mathcal{E}}$ *imply that* $Im(\phi^{\mathcal{E}}) \subseteq \mathcal{E}_J$*. Indeed every element* $(p,\mathcal{L},e,R)$ *in the image of* $\phi^{\mathcal{E}}$ *is such that* $(p,\mathcal{L},e)$ *is an* α*-equilibrium, and* $(p,\mathcal{L},R)$ *is such that* $< V(p,R) > \subseteq \mathcal{L}$*. Next,* $Im(\phi^{\mathcal{E}}) \supseteq \mathcal{E}_J$ *follows from the observation that* $\phi^{\mathcal{E}} \circ \tau^{\mathcal{E}} = id_{\mathcal{E}}$*. We conclude that* $\mathcal{E}_J = \phi^{\mathcal{E}}(\mathbb{P}\times\mathbb{N}_J\times\widehat{\Omega}\times\mathbb{R}^{SJ(L-1)})$*, is a smooth submanifold of* $\mathbb{P}\times\mathsf{G}^{J,S}\times\Omega(\omega)\times\mathbb{R}^{SJL}$ *diffeomorphic to* $\mathbb{P}\times\mathbb{N}_J\times\widehat{\Omega}\times\mathbb{R}^{SJ(L-1)}$ *through* $\phi^{\mathcal{E}}$*.* ■

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Summary of Mathematical Notation

$\mathsf{G}^{J,S}$ *is the Grassmanian manifold of* J*-planes in* $\mathbb{R}^S$

$\upsilon = \left\{\mathcal{L}, y \in \mathsf{G}^{J,S} \times \mathbb{R}^S : y \in \mathcal{L}\right\}$ *is the canonical vector bundle over* $\mathsf{G}^{J,S}$

$\upsilon^{\perp}$ *is the orthogonal complement of* υ

ε *is the trivial vector bundle*

$\oplus$ *is the Whitney sum*

$\cong$ *is an homeomorphism relation*

$\square$ *tensor product of vectors: for any* $x \in \mathbb{R}^S$, $y \in \mathbb{R}^{SL}$, $x \square y = \left(...., x_s\left(y_{s1}, ..y_{sl}, .., y_{sL}\right), ...\right) \in \mathbb{R}^{SL}$

$\dot{\cup}$ *denotes the disjoint union of sets*